Off–Diagonal Ekpyrotic Scenarios and Equivalence of Modified, Massive and/or Einstein Gravity

Sergiu I. Vacaru *

Rector's Office, Alexandru Ioan Cuza University, Alexandru Lapuşneanu street, nr. 14, UAIC – Corpus R, office 323; Iaşi, Romania, 700057

April 23, 2013

Abstract

We show how generic off–diagonal cosmological solutions depending, in general, on all spacetime coordinates can be constructed in massive gravity using the anholonomic frame deformation method. There are found new classes of locally anisotropic and (in) homogeneous cosmological metrics with open and closed spatial geometries. Such solutions describe the late time acceleration due to effective cosmological terms induced by nonlinear off–diagonal interactions and graviton mass. The cosmological metrics and related Stückelberg fields are constructed in explicit form up to nonholonomic frame transforms of the Friedmann–Lamaître–Robertson–Walker (FLRW) coordinates. The solutions include matter, graviton mass and other effective sources modelling nonlinear gravitational and matter fields interactions with polarization of physical constants and deformations of metrics, which may explain certain dark energy and dark matter effects. There are stated the conditions when such configurations mimic interesting solutions in general relativity and modifications and recast the general Painlevé–Gullstrand and FLRW metrics. Finally, we sketch a reconstruction procedure for a subclass of off–diagonal cosmological solutions which describe cyclic and ekpyrotic universes.

Keywords: massive gravity, modified gravity, off–diagonal cosmological solutions; ekpyrotic and little rip universe.

PACS: 04.50.Kd, 04.90.+e, 98.80.Jk, 98.80.Cq, 95.30.Sf, 95.36.+x, 95.35.+d

*sergiu.vacaru@uaic.ro;
http://www.uaic.ro/uaic/bin/view/Research/AdvancedTheoretical

The idea that graviton may have a nontrivial mass was proposed by Fierz and Pauli work [1] (for recent reviews and related f(R) modifications, see [2]). The key steps in elaborating a modern version of a ghost free (bimetric) massive gravity theory were made in a series of papers: The so–called vDVZ discontinuity problem was solved using the Vainshtein mechanism [3] (avoiding discontinuity by going beyond the linear theory), or following more recent approaches based on DGP model [4]. But none solution was found for another problem with ghosts because at nonlinear order in massive gravity appears a sixth scalar degree of freedom as a ghost (see the Boulware and Deser paper and similar issues related to the effective field theory approach in Refs. [5]). That stoppled for almost two decades the research on formulating a consistent theory of massive gravity.

Recently, a substantial progress was made when de Rham and co–authors had shown how to eliminate the scalar mode and Hassan and Rosen established a complete proof for a class of bigravity / bimetric gravity theories, see [6]. The second metric describes an effective exotic matter related to massive gravitons and does not suffer from ghost instability to all orders in a perturbation theory and away from the decoupling limit.

The possibility that the graviton has a nonzero mass $\mathring{\mu}$ results not only in fundamental theoretical implications but give rise to straightforward phenomenological consequences. For instance, a gravitational potential of Yukawa form $\sim e^{-\mathring{\mu}r}/r$ results in decay of gravitational interactions at scales $r \geq \mathring{\mu}^{-1}$ and this can result in the accelerated expansion of the Universe. This way, a theory of massive gravity provides alternatives to dark energy and, via effective polarizations of fundamental physical constants (in result of generic off–diagonal nonlinear interactions), may explain certain dark matter effects. Recently, various cosmological models derived for ghost free (modified) massive gravity and bigravity theories have been elaborated and studied intensively (see, for instance, Refs. [7, 2, 8]).

The goal of this work is to construct generic off–diagonal cosmological solutions in massive gravity theory, MGT, and state the conditions when such configurations are modelled equivalently in general relativity (GR). As a first step, we consider off–diagonal deformations of a "prime" cosmological solution taken in general Painlevé–Gullstrand (PG) form, when the Friedman–Lamaître–Robertson–Walker (FLRW) can be recast for well–defined geometric conditions. At the second step, the "target" metrics will be generated to possess one Killing symmetry (or other none Killing symmetries) and depend on timelike and certain (all) spacelike coordinates. In general, such off–diagonal solutions are with local anisotropy and inhomogeneities for effective cosmological constants and polarizations of other physical constants and coefficients of cosmological metrics which can be modelled both in MGT and GR. Finally (the third step), we shall emphasize and speculate on importance of off–diagonal nonlinear gravitational interactions for elaborating cosmological scenarios when dark matter and dark energy effects can be

explained by anisotropic polarizations of vacuum and/or de Sitter like configurations.

We study modified massive gravity theories determined on a pseudo–Riemannian spacetime $\mathbf{V}$ with physical metric $\mathbf{g} = \{\mathbf{g}_{\mu\nu}\}$ and certain fiducial metric as we shall explain below. The action for our model is

$$S = \frac{1}{16\pi}\int \delta u^4 \sqrt{|\mathbf{g}_{\alpha\beta}|}[\widehat{f}(\widehat{R}) - \frac{\mathring{\mu}^2}{4}\mathcal{U}(\mathbf{g}_{\mu\nu}, \mathbf{K}_{\alpha\beta}) + \ {}^{m}L] \quad (1)$$
$$= \frac{1}{16\pi}\int \delta u^4 \sqrt{|\mathbf{g}_{\alpha\beta}|}[f(R) + \ {}^{m}L]. \quad (2)$$

In this formula, $\widehat{R}$ is the scalar curvature for an auxiliary (canonical) connection $\widehat{\mathbf{D}}$ uniquely determined by two conditions 1) it is metric compatible, $\widehat{\mathbf{D}}\mathbf{g} = 0$, and 2) its h- and v-torsions are zero (but there are nonzero $h-v$ components of torsion $\widehat{\mathcal{T}}$ completely determined by $\mathbf{g}$) for a conventional splitting $\mathbf{N}:\ T\mathbf{V} = h\mathbf{V}\oplus v\mathbf{V}$, see details in [9].[1] The "priority" of the connection $\widehat{\mathbf{D}}$ is that it allows to decouple the field equations in various gravity theories and construct exact solutions in very general forms. We shall work with generic off–diagonal metrics and generalized connections depending on all spacetime coordinates when, for instance, of type $\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}[\widehat{\mathcal{T}}]$. Such distortion relations from the Levi–Civita (LC) connection ∇ are uniquely determined by a distorting tensor $\widehat{\mathbf{Z}}$ completely defined by $\widehat{\mathcal{T}}$ and (as a consequence for such models) by $(\mathbf{g}, \mathbf{N})$. Having constructed integral varieties (for instance, locally anisotropic and/or inhomogeneous cosmological ones), we can impose additional nonholonomic (non–integrable constraints) when $\widehat{\mathbf{D}}_{|\widehat{\mathcal{T}}=0} \to \nabla$ and $\widehat{R} \to R$, where R is the scalar curvature of ∇, and it is possible to extract exact solutions in GR.

The theories with actions of type (1) generalize the so–called modified $f(R)$ gravity, see reviews and original results in [2], and the ghost–free massive gravity (by de Rham, Gabadadze and Tolley, dRGT) [6]. We use the units when $\hbar = c = 1$ and the Planck mass M_{Pl} is defined via $M_{Pl}^2 = 1/8\pi G$ with 4–d Newton constant G. We write δu^4 instead of d^4u because there are used N–elongated differentials (see below the formulas (11)) and consider $\mathring{\mu} = const$ as the mass of graviton. For LC–configurations, we can fix (as a particular case) condition of type

$$\widehat{f}(\widehat{R}) - \frac{\mathring{\mu}^2}{4}\mathcal{U}(\mathbf{g}_{\mu\nu}, \mathbf{K}_{\alpha\beta}) = f(\widehat{R}), \quad \text{or} \quad \widehat{f}(\widehat{R}) = f(R), \quad \text{or} \quad \widehat{f}(\widehat{R}) = R, \quad (3)$$

which depend on the type of models we elaborate and what classes of solutions we wont to construct. It will be possible to find solutions in explicit form if we fix the coefficients $\{N_i^a\}$ of $\mathbf{N}$ and local frames for $\widehat{\mathbf{D}}$ when

[1] We consider a conventional $2+2$ splitting when coordinates are labelled in the form $u^\alpha = (x^i, y^a)$, or $u = (x, y)$, with indices $i, j, k... = 1, 2$ and $a, b, ... = 3, 4$. There will be used boldface symbols in order to emphasize that certain geometric/physical objects and/or formulas are written with respect to N–adapted bases (11). There will be considered also left up/low indices as labels for some geometric/physical objects.

$\widehat{R} = const$ and $\partial_\alpha \widehat{f}(\widehat{R}) = (\partial_{\widehat{R}}\widehat{f}) \times \partial_\alpha \widehat{R} = 0$ but, in general, $\partial_\alpha f(R) \neq 0$. The equations of motion for such modified massive gravity theory can be written[2]

$$(\partial_{\widehat{R}}\widehat{f})\widehat{\mathbf{R}}_{\mu\nu} - \frac{1}{2}\widehat{f}(\widehat{R})\mathbf{g}_{\mu\nu} + \mathring{\mu}^2 \mathbf{X}_{\mu\nu} = M_{Pl}^{-2}\mathbf{T}_{\mu\nu}, \quad (4)$$

where M_{Pl} is the Plank mass, $\widehat{\mathbf{R}}_{\mu\nu}$ is the Einstein tensor for a pseudo–Riemannian metric $\mathbf{g}_{\mu\nu}$ and $\widehat{\mathbf{D}}$, $\mathbf{T}_{\mu\nu}$ is the standard matter energy–momentum tensor. For $\widehat{\mathbf{D}} \to \nabla$, we get $\widehat{\mathbf{R}}_{\mu\nu} \to R_{\mu\nu}$ with a standard Ricci tensor $R_{\mu\nu}$ for ∇. The effective energy–momentum tensor $\mathbf{X}_{\mu\nu}$ is defined in a "sophisticate" form by the potential of graviton $\mathcal{U} = \mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4$, where α_3 and α_4 are free parameters. The values $\mathcal{U}_2, \mathcal{U}_3$ and $\mathcal{U}_4$ are certain polynomials on traces of some other polynomials of a matrix $\mathcal{K}_\mu^\nu = \delta_\mu^\nu - \left(\sqrt{g^{-1}\Sigma}\right)_\mu^\nu$ for a tensor determined by four Stückelberg fields $\phi^{\underline{\mu}}$ as

$$\Sigma_{\mu\nu} = \partial_\mu \phi^{\underline{\mu}} \partial_\nu \phi^{\underline{\nu}} \eta_{\underline{\mu\nu}}, \quad (5)$$

when $\eta_{\underline{\mu\nu}} = (1, 1, 1, -1)$. Following a series of arguments presented in [8], when the parameter choice $\alpha_3 = (\alpha - 1)/3, \alpha_4 = (\alpha^2 - \alpha + 1)/12$ is useful for avoiding potential ghost instabilities, we can fix

$$\mathbf{X}_{\mu\nu} = \alpha^{-1}\mathbf{g}_{\mu\nu}. \quad (6)$$

De Sitter solutions for an effective cosmological constant are possible, for instance, for ansatz of PG type,

$$ds^2 = U^2(r,t)[dr + \epsilon\sqrt{f(r,t)}dt]^2 + \widetilde{\alpha}^2 r^2(d\theta^2 + \sin^2\theta d\varphi^2) - V^2(r,t)dt^2. \quad (7)$$

In above formula, there are used spherical coordinates labelled in the form $u^\beta = (x^1 = r, x^2 = \theta, y^3 = \varphi, y^4 = t)$, the function f takes non–negative values and the constant $\widetilde{\alpha} = \alpha/(\alpha + 1)$ and $\epsilon = \pm 1$. For such bimetric configurations, the Stückelberg fields are parameterized in the unitary gauge as $\phi^{\underline{4}} = t$ and $\phi^{\underline{1}} = r\widehat{n}^{\underline{1}}, \phi^{\underline{2}} = r\widehat{n}^{\underline{2}}, \phi^{\underline{3}} = r\widehat{n}^{\underline{3}}$, where a three dimensional (3–d) unit vector is defined as $\widehat{n} = (\widehat{n}^{\underline{1}} = \sin\theta\cos\varphi, \widehat{n}^{\underline{2}} = \sin\theta\sin\varphi, \widehat{n}^{\underline{3}} = \cos\theta)$. Any PG metric of type (7) defines solutions both in GR and in MGT. It allows us to extract the de Sitter solution, in the absence of matter, and to obtain standard cosmological equations with FLRW metric, for a perfect fluid source

$$T_{\mu\nu} = [\rho(t) + p(t)]\, u_\mu u_\nu + p(t) g_{\mu\nu}, \quad (8)$$

where $u_\mu = (0, 0, 0, -V)$ can be reproduced for the effective cosmological constant ${}^{eff}\lambda = \mathring{\mu}^2/\alpha$. It is also possible to express metrics of type (7) in a familiar cosmological FLRW form (see formulas (23), (24) and (27) in [8]).

[2]see details on action and variational methods in [6]; we shall follow some conventions from [8]; the Einstein summation rule on repeating indices will be applied if the contrary is not stated

Let us consider an ansatz

$$
\begin{aligned}
ds^2 &= \eta_1(r,\theta)\mathring{g}_1(r)dr^2 + \eta_2(r,\theta)\mathring{g}_2(r)d\theta^2 \\
&+\omega^2(r,\theta,\varphi,t)\{\eta_3(r,\theta,t)\mathring{h}_3(r,\theta)[d\varphi + n_i(r,\theta)dx^i]^2 \\
&+\eta_4(r,\theta,t)\mathring{h}_4(r,\theta,t)[dt + (w_i(x^k,t) + \mathring{w}_i(x^k))dx^i]^2\},
\end{aligned} \tag{9}
$$

with Killing symmetry on $\partial_3 = \partial_\varphi$, which (in general) can not be diagonalized by coordinate transforms. The values η_α are called "polarization" functions; ω is the so–called "vertical", v, conformal factor. The off–diagonal, N–coefficients, are labelled $N_i^a(x^k, y^4)$, where (for this ansatz) $N_i^3 = n_i(r,\theta)$ and $N_i^4 = w_i(x^k,t) + \mathring{w}_i(x^k)$. The data for the "primary" metric are

$$
\begin{aligned}
\mathring{g}_1(r) &= U^2 - \mathring{h}_4(\mathring{w}_1)^2, \mathring{g}_2(r) = \widetilde{\alpha}^2 r^2, \mathring{h}_3 = \widetilde{\alpha}^2 r^2 \sin^2\theta, \mathring{h}_4 = \sqrt{|fU^2 - V^2|}, \\
\mathring{w}_1 &= \epsilon\sqrt{f}U^2/\mathring{h}_4, \mathring{w}_2 = 0, \mathring{n}_i = 0,
\end{aligned} \tag{10}
$$

when the coordinate system is such way fixed that the values f, U, V in (7) result in a coefficient $\mathring{g}_1$ depending only on r.

We shall work with respect to a class of N–adapted (dual) bases

$$
\begin{aligned}
\mathbf{e}_\alpha &= (\mathbf{e}_i = \partial_i - N_i^b\partial_b, e_a = \partial_a = \partial/\partial y^a) \text{ and} \\
\mathbf{e}^\beta &= (e^j = dx^i, \mathbf{e}^b = dy^b + N_c^b dy^c),
\end{aligned} \tag{11}
$$

which are nonholonomic (equivalently, anholonomic) because, in general, there are satisfied relations of type $\mathbf{e}_\alpha\mathbf{e}_\beta - \mathbf{e}_\beta\mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma$, for certain non-trivial anholonomy coefficients $W_{\alpha\beta}^\gamma(u)$. For simplicity, we shall consider energy momentum sources (8) and effective (6) which up to frame/coordinate transforms can be parameterized in the form

$$
\Upsilon_\beta^\alpha = \frac{1}{M_{Pl}^2(\partial_{\widehat{R}}\widehat{f})}(\mathbf{T}_\beta^\alpha + \alpha^{-1}\mathbf{X}_\beta^\alpha) = \frac{1}{M_{Pl}^2(\partial_{\widehat{R}}\widehat{f})}({}^mT + \alpha^{-1})\delta_\beta^\alpha = ({}^m\Upsilon + {}^\alpha\Upsilon)\delta_\beta^\alpha \tag{12}
$$

for constant values ${}^m\Upsilon := M_{Pl}^{-2}(\partial_{\widehat{R}}\widehat{f})^{-1}\, {}^mT$ and ${}^\alpha\Upsilon = M_{Pl}^{-2}(\partial_{\widehat{R}}\widehat{f})^{-1}\alpha^{-1}$, with respect to N–adapted frames (11).[3]

Let us explain an important decoupling property of the gravitational field equations in GR and various generalizations/ modifications studied in details in Refs. [9]. That anholonomic frame deformation method (AFDM) can be applied for decoupling, and constructing solutions of the MGT field equations (4) with any effective source (12). We consider target off–diagonal metrics $\mathbf{g} = (g_i = \eta_i\mathring{g}_i, h_a = \eta_a\mathring{h}_a, N_j^a)$ [there is not summation on repeating indices in this formula] with coefficients determined by ansatz (9). For convenience, we shall use brief denotations for partial derivatives: $\partial_1\psi =$

[3] In general, such sources are not diagonal and may depend on spacetime coordinates. We fix such N–adapted parameterizations which will allow to construct exact solutions in explicit form.

$\psi^{\bullet}, \partial_2\psi = \psi', \partial_3\psi = \psi^{\diamond}$ and $\partial_4\psi = \psi^*$. Computing the N–adapted coefficients of the Ricci and Einstein tensors (see details in Refs. [9]), we transform (4) into a system of nonlinear partial differential equations (PDE):

$$\begin{aligned}
\widehat{R}^1_1 &= \widehat{R}^2_2 \Longrightarrow \psi^{\bullet\bullet} + \psi'' = 2(\,^m\Upsilon + \,^\alpha\Upsilon), \qquad (13)\\
\widehat{R}^3_3 &= \widehat{R}^4_4 \Longrightarrow \phi^* h_3^* = 2h_3h_4(\,^m\Upsilon + \,^\alpha\Upsilon),\\
\widehat{R}_{3k} &\Longrightarrow n_i^{**} + \gamma n_i^* = 0,\ \widehat{R}_{4k} \Longrightarrow \beta w_i - \alpha_i = 0,\\
\partial_k\omega &= n_k\omega^{\diamond} + w_k\omega^*, \qquad (14)
\end{aligned}$$

$$\text{for } \phi = \ln\left|\frac{h_3^*}{\sqrt{|h_3h_4|}}\right|, \gamma := (\ln\frac{|h_3|^{3/2}}{|h_4|})^*, \quad \alpha_i = \frac{h_3^*}{2h_3}\partial_i\phi,\ \beta = \frac{h_3^*}{2h_3}\phi^*, \qquad (15)$$

where $\Longrightarrow$ is used in order to show that certain equations follow from respective coefficients of the Ricci tensor $\widehat{\mathbf{R}}_{\mu\nu}$. In these formulas, the system of coordinates and polarization functions are fixed for configurations with $g_1 = g_2 = e^{\psi(x^k)}$ and nonzero values ϕ^* and h_a^*. The equations result in solutions for the Levi–Civita configurations (with zero torsion) if the coefficients of metrics are subjected to the conditions

$$w_i^* = \mathbf{e}_i \ln\sqrt{|\ h_4|}, \mathbf{e}_i \ln\sqrt{|\ h_3|} = 0, \partial_i w_j = \partial_j w_i \text{ and } n_i^* = 0. \qquad (16)$$

The system of nonlinear PDE (13)–(16) can be integrated in general forms for any ω constrained by a system of linear first order equations (14). The explicit solutions are given by quadratic elements

$$\begin{aligned}
ds^2 &= e^{\psi(x^k)}[(dx^1)^2 + (dx^2)^2] + \frac{\Phi^2\omega^2}{4\,(\,^m\Upsilon + \,^\alpha\Upsilon)}\mathring{h}_3[d\varphi + (\partial_k\ n)\,dx^k]^2\\
&\quad - \frac{(\Phi^*)^2\omega^2}{(\,^m\Upsilon + \,^\alpha\Upsilon)\Phi^2}\mathring{h}_4[dt + (\partial_i\ \widetilde{A})dx^i]^2. \qquad (17)
\end{aligned}$$

for any $\Phi = \check{\Phi}$, $(\partial_i\check{\Phi})^* = \partial_i\check{\Phi}^*$ and $w_i + \mathring{w}_i = \partial_i\check{\Phi}/\check{\Phi}^* = \partial_i\ \widetilde{A}$.[4] To generate new solutions we can consider arbitrary nontrivial sources, $\,^m\Upsilon + \,^\alpha\Upsilon \neq 0$, and generating functions, $\Phi(x^k,t) := e^{\phi}$ and $n_k = \partial_k n(x^i)$. Such metrics are generic off–diagonal and can not be diagonalized via coordinate transforms in a finite spacetime region because, in general, the anholonomy coefficients $W^{\gamma}_{\alpha\beta}$ for (11) are not zero (we can check by explicit computations). The polarization η–functions for (17) are computed in the form

$$\eta_1 = e^{\psi}/\mathring{g}_1, \eta_2 = e^{\psi}/\mathring{g}_2, \eta_3 = \Phi^2/4\,(\,^m\Upsilon + \,^\alpha\Upsilon), \eta_4 = (\Phi^*)^2/(\,^m\Upsilon + \,^\alpha\Upsilon)\Phi^2. \quad (18)$$

[4] We can construct exact solutions even such conditions are not satisfied, i.e. the zero torsion conditions are not stated or there are given in non–explicit form; this way, it is possible to generate off–diagonal metrics and nonholonomically induced torsions etc, see details in [9]. There will pe presented physical arguments for what type of generating/ integration functions and sources we have to chose in order to construct realistic scenarios for Universe acceleration and observable dark energy/ matter effects.

So, prescribing any generating functions $\check{\Phi}(r,\theta,t)$, $n(r,\theta)$, $\omega(r,\theta,\varphi,t)$ and sources ${}^{m}\Upsilon$, ${}^{\alpha}\Upsilon$ and then computing $\check{A}(r,\theta,t)$, we can transform any PG (and FLRW) metric $\mathbf{\mathring{g}}=(\mathring{g}_i,\mathring{h}_a,\mathring{w}_i,\mathring{n}_i)$ in MGT and/or GR into new classes of generic off–diagonal exact solutions depending on all spacetime coordinates. Such metrics define Einstein manifolds in GR with effective cosmological constants ${}^{m}\Upsilon+{}^{\alpha}\Upsilon$. With respect to N–adapted frames (11) the coefficients of metric encode contributions from massive gravity, determined by ${}^{\alpha}\Upsilon$, and matter fields, included in ${}^{m}\Upsilon$.

It is possible to provide an "alternative" treatment of (17) as exact solutions in MGT. Following such an approach, we have to define and analyze the properties of fiducial Stückelberg fields $\phi^{\underline{\mu}}$ and the corresponding bimetric structure resulting in target solutions $\mathbf{g}=(g_i,h_a,N_j^a)$: Let us analyze the primary configurations related to $\mathring{\phi}^{\underline{\mu}}=(\mathring{\phi}^{\underline{i}}=a(\tau)\rho\widetilde{\alpha}^{-1}\widehat{n}^{\underline{i}},\mathring{\phi}^{\underline{3}}=a(\tau)\rho\widetilde{\alpha}^{-1}\widehat{n}^{\underline{3}},\mathring{\phi}^{\underline{4}}=\tau\kappa^{-1})$, when the corresponding prime PG–metric $\mathbf{\mathring{g}}$ is taken in FLRW form $ds^2=a^2(d\rho^2/(1-K\rho^2)+\rho^2(d\theta^2+\sin^2\theta d\varphi^2))-d\tau^2$. The related fiducial tensor (5) is computed

$$\mathring{\Sigma}_{\underline{\mu}\underline{\nu}}du^{\underline{\mu}}du^{\underline{\nu}}=\frac{a^2}{\tilde{\alpha}^2}[d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\varphi^2)+2H\rho d\rho d\tau-(\frac{\tilde{\alpha}^2}{\kappa^2a^2}-H^2\rho^2)d\tau^2],$$

where the coefficients and coordinates are re–defined in the form $r\to\rho=\widetilde{\alpha}r/a(\tau)$ and $t\to\tau=\kappa t$, for $K=0,\pm1;\kappa$ is an integration constant; $H:=d\ln a/d\tau$ and the local coordinates are parameterized in the form $x^{\underline{1}}=\rho,x^{\underline{2}}=\theta,y^{\underline{3}}=\varphi,y^{\underline{4}}=\tau$.

For a target metric $\mathbf{g}=\mathbf{g}_{\alpha\beta}$ and frames $\mathbf{e}_\alpha=\mathbf{e}_\alpha^{\ \underline{\alpha}}\partial_{\underline{\alpha}}$, we can write

$$\mathbf{g}_{\alpha\beta}=\mathbf{e}_\alpha^{\ \underline{\alpha}}\mathbf{e}_\beta^{\ \underline{\beta}}\eta_{\underline{\alpha}\underline{\beta}}=\left[\begin{array}{cc} g_{ij}+N_i^aN_j^bg_{ab} & N_i^ah_{ab}\\ N_i^ah_{ab} & h_{ab}\end{array}\right],\ \text{for } \mathbf{e}_\alpha^{\ \underline{\alpha}}=\left[\begin{array}{cc}\mathbf{e}_i^{\ \underline{i}} & N_i^b\mathbf{e}_b^{\ \underline{a}}\\ 0 & \mathbf{e}_a^{\ \underline{a}}\end{array}\right].$$

The values $g_{ij}=\mathbf{e}_i^{\ \underline{\alpha}}\mathbf{e}_j^{\ \underline{\beta}}\eta_{\underline{\alpha}\underline{\beta}}=e^{\psi}\delta_{ij}=diag[\eta_i\mathring{g}_i]$, $h_{ab}=\mathbf{e}_i^{\ \underline{\alpha}}\mathbf{e}_j^{\ \underline{\beta}}\eta_{\underline{\alpha}\underline{\beta}}=diag[\eta_a\mathring{h}_a],N_i^3=\partial_k\ n$ and $N_i^4=\partial_k\ \widetilde{A}$ are related algebraically to data (18) resulting in off–diagonal solutions (17). We can compute the "target" Stückelberg fields as $\phi^{\mu'}=\mathbf{e}^{\mu'}_{\ \underline{\mu}}\phi^{\underline{\mu}}$ with $\mathbf{e}^{\mu'}_{\ \underline{\mu}}$ being inverse to $\mathbf{e}_\alpha^{\ \underline{\alpha}}$, and the fiducial tensor $\Sigma_{\alpha\beta}=(\mathbf{e}_\alpha\phi^{\underline{\mu}})(\mathbf{e}_\beta\phi^{\underline{\nu}})\eta_{\underline{\mu}\underline{\nu}}=\mathbf{e}_\alpha^{\ \underline{\alpha}}\mathbf{e}_\beta^{\ \underline{\beta}}\Sigma_{\underline{\alpha}\underline{\beta}}$. If the value $\mathring{\Sigma}_{\underline{\mu}\underline{\nu}}$ carries information about two constants κ and $\widetilde{\alpha}$, a tensor $\Sigma_{\mu\nu}$ associated to off–diagonal solutions encodes data about generating and integration functions and via superpositions on possible Killing symmetries, on various integration constants [10]. In the framework of MGT, two cosmological solutions $\mathbf{\mathring{g}}$ and $\mathbf{g}$ related by nonholonomic deformations[5] are characterised respectively by two invariants $\mathring{I}^{\underline{\alpha}\underline{\beta}}=\mathring{g}^{\alpha\beta}\partial_\alpha\mathring{\phi}^{\underline{\alpha}}\partial_\beta\mathring{\phi}^{\underline{\beta}}$ and $\mathbf{I}^{\alpha\beta}=\mathbf{g}^{\alpha\beta}\mathbf{e}_\alpha\phi^{\underline{\alpha}}\mathbf{e}_\beta\phi^{\underline{\beta}}$. The tensor $\mathring{I}^{\underline{\alpha}\underline{\beta}}$ does not contain singularities because there are not coordinate singularities on horizon for PG metrics. The symmetry of $\Sigma_{\mu\nu}$ is not the same as that

[5]involving not only frame transforms but also deformation of the linear connection structure when at the end there are imposed additional constraints for zero torsion

of $\mathring{\Sigma}_{\underline{\mu\nu}}$ and the singular behaviour of $\mathbf{I}^{\alpha\beta}$ depends on the class of generating and integration functions we chose of constructing a target solution $\mathbf{g}$.

In GR and/or Einstein–Finsler gravity theories [9], off–cosmological solutions of type (17) were found to generalized various models of Biachi, Kasner, Gödel and other universes. For instance, Bianchi type anisotropic cosmological metrics are generated if we impose corresponding Lie algebra symmetries on metrics. It was emphasized in [8] that "any PG–type solution in general relativity (with a cosmological constant) is also a solution to massive gravity." Such a conclusion can be extended to a large class of generic off–diagonal cosmological solutions generated by effective cosmological constants but it is not true, for instance, if we consider nonholonomic deformations with nonholonomically induced torsion like in metric compatible Finsler theories.

We note that the analysis of cosmological perturbations around an off–diagonal cosmological background is not trivial because the fiducial and reference metrics do not respect the same symmetries. Nevertheless, fluctuations around de Sitter backgrounds seem to have a decoupling limit which implies that one can avoid potential ghost instabilities if the parameter choice is considered both for diagonal and off–diagonal cosmological solutions, see details in [11]. This special choice also allows us to have a structure $X_{\mu\nu} \sim g_{\mu\nu}$ at list in N–adapted frames when the massive gravity effects can be approximated by effective cosmological constants and exact solutions in MGT which are also solutions in GR.

Let us consider three examples of off–diagonal cosmological solutons with solitonic modifications in MGT and (with alternative interpretation) GR. Two and three dimensional solitonic waves are typical nonlinear wave configurations which can be used for generating spacetime metrics with Killing, or non–Killing, symmetries and can be characterised by additional parametric dependencies and solitonic symmetries.

Example 1: Taking a nonlinear radial (solitonic, with left s-label) generating function

$$\Phi = {}^{s}\check{\Phi}(r,t) = 4 \arctan e^{q\sigma(r-vt)+q_0} \tag{19}$$

and $\omega = 1$, we construct a metric

$$\begin{aligned} \mathbf{ds}^2 &= e^{\psi(r,\theta)}(dr^2 + d\theta^2) + \frac{{}^{s}\check{\Phi}^2}{4\,(\,{}^{m}\Upsilon + {}^{\alpha}\Upsilon)} \mathring{h}_3(r,\theta) d\varphi^2 \\ & - \frac{(\partial_t\, {}^{s}\check{\Phi})^2}{(\,{}^{m}\Upsilon + {}^{\alpha}\Upsilon)\, {}^{s}\check{\Phi}^2} \mathring{h}_4(r,t)[dt + (\partial_r\, \widetilde{A})dr]^2, \end{aligned} \tag{20}$$

where, for simplicity, we consider $n(r,\theta) = 0$, $\widetilde{A}(r,t)$ is defined as a solution of ${}^{s}\check{\Phi}^{\bullet}/\, {}^{s}\check{\Phi}^{*} = \partial_r\, \widetilde{A}$ and $\mathring{h}_a$ are given by PG–data (10). The generating function (19), where $\sigma^2 = (1-v^2)^{-1}$ for constants q, q_0, v, is just a 1–soliton solution of the sine–Gordon equation ${}^{s}\check{\Phi}^{**} - {}^{s}\check{\Phi}^{\bullet\bullet} + \sin\, {}^{s}\check{\Phi} = 0$. For any

class of small polarizations with $\eta_a \sim 1$), we can consider that the source $({}^m\Upsilon + {}^\alpha\Upsilon)$ is polarized by ${}^s\check{\Phi}^{-2}$ when $h_3 \sim \mathring{h}_3$ and $h_4 \sim \mathring{h}_4({}^s\check{\Phi}^*)^2/{}^s\check{\Phi}^{-4}$ with an off–diagonal term $\partial_r \widetilde{A}$ resulting in a stationary solitonic universe. If we consider that $(\partial_{\widehat{R}}\widehat{f})^{-1} = {}^s\check{\Phi}^{-2}$ in (12), we can model $\widehat{f}$–interactions of type (1) via off–diagonal interactions and "gravitational polarizations". In absence of matter, ${}^m\Upsilon = 0$, the off–diagonal cosmology is completely determined by ${}^\alpha\Upsilon$ when ${}^s\check{\Phi}$ transforms $\mathring{\mu}$ into an anisotropically polarized/variable mass of solitonic waves. Such configurations can be modelled if ${}^m\Upsilon \ll {}^\alpha\Upsilon$. If ${}^m\Upsilon \gg {}^\alpha\Upsilon$, we generate cosmological models determined by distribution off matter fields when contributions from massive gravity are with small anisotropic polarization. For a class of nonholonomic constraints on Φ and ψ (which may be not of solitonic type), when solutions (20) are of type (9) with $\eta_\alpha \sim 1$ and $n_i, w_i \sim 0$, we approximate PG–metrics of type (7). Hence, by an appropriate choice of generating functions and sources, we can model equivalently modified gravity effects, massive gravity contributions or matter field configurations in GR and MGT interactions. For well defined conditions, such configurations can be studied in the framework of some classes of off–diagonal solutions in Einstein gravity with effective cosmological constants.

Example 2: Three dimensional solitonic anistoropic waves can be generated, for instance, if we take instead of (19) a generating functions ${}^s\check{\Phi}(r,\theta,t)$ which is a solution of the Kadomtsev–Petivashvili, KdP, equations [12],

$$\pm\, {}^s\check{\Phi}'' + ({}^s\check{\Phi}^* + {}^s\check{\Phi}\ {}^s\check{\Phi}^\bullet + \epsilon\, {}^s\check{\Phi}^{\bullet\bullet\bullet})^\bullet = 0,$$

when solutions induce certain anisotropy on θ.[6] In the dispersionless limit $\epsilon \to 0$, we can consider that the solutions are independent on θ and determined by Burgers' equation ${}^s\check{\Phi}^* + {}^s\check{\Phi}\ {}^s\check{\Phi}^\bullet = 0$. The solutions can be parameterized and treated similarly to (20) but with, in general, a nontrivial term $(\partial_\theta \widetilde{A})d\theta$ after $\mathring{h}_4$, when ${}^s\check{\Phi}^\bullet/{}^s\check{\Phi}^* = \widetilde{A}^\bullet$ and ${}^s\check{\Phi}'/{}^s\check{\Phi}^* = \widetilde{A}'$.

Example 3: Solitonic waves can be considered for a nontrivial vertical conformal v–factor as in (9), for instance, of KdP type, when $\omega = \breve{\omega}(r,\varphi,t)$, when $x^1 = r$, $x^2 = \theta, y^3 = \varphi, y^4 = t$, for

$$\pm\, \breve{\omega}^{\diamond\diamond} + (\partial_t \breve{\omega} + \breve{\omega}\ \breve{\omega}^\bullet + \epsilon \breve{\omega}^{\bullet\bullet\bullet})^\bullet = 0, \qquad (21)$$

In the dispersionless limit $\epsilon \to 0$, we can consider that the solutions are independent on angle φ and determined by Burgers' equation $\breve{\omega}^* + \breve{\omega}\ \breve{\omega}^\bullet = 0$. The conditions (14) impose an additional constraint $\mathbf{e}_1\breve{\omega} = \breve{\omega}^\bullet + w_1(r,\theta,\varphi)\breve{\omega}^* + n_1(r,\theta)\breve{\omega}^\diamond = 0$. In the system of coordinates when $\breve{\omega}' = 0$, we can fix $w_2 = 0$

[6] In a similar form, we can construct various types of vacuum gravitational 2-d and 3-d configurations characterized by solitonic hierarchies and related bi–Hamilton structures, for instance, of KdP equations with possible mixtures with solutions for 2-d and 3-d sine–Gordon equations etc, see details in Ref. [13].

and $n_2 = 0$. For any arbitrary generating function with LC–configuration, $\check{\Phi}(r,\theta,t)$, we construct exact solutions

$$\begin{aligned}\mathbf{ds}^2 &= e^{\psi(r,\theta)}(dr^2 + d\theta^2) + \frac{\check{\Phi}^2\breve{\omega}^2}{4\ (\ {}^m\Upsilon + \ {}^\alpha\Upsilon)}\mathring{h}_3(r,\theta)d\varphi^2 \\ &- \frac{(\partial_t\ \check{\Phi})^2\breve{\omega}^2}{(\ {}^m\Upsilon + \ {}^\alpha\Upsilon)\ \check{\Phi}^2}\mathring{h}_4(r,t)[dt + (\partial_r\ \widetilde{A})dr]^2, \end{aligned} \qquad (22)$$

which are generic off–diagonal and depend on all spacetime coordinates. Such stationary cosmological solutions are with polarizations on two angles θ and φ. Nevertheless, the character of anisotropies is different for metrics of type (20) and (22). In the third class of metrics, we obtain a Killing symmetry on ∂_φ only in the limit $\breve{\omega} \to 1$, but in the first two ones, such a symmetry exists generically. For (22), the value $\check{\Phi}$ is not obligatory a solitonic one which can be used for additional off–diagonal modifications of solutions and various types of polarizations. We can provide an interpretation similar to that in Example 1, if the generating and integration functions are chosen to satisfy the conditions $\eta_\alpha \sim 1$ and $n_i, w_i \sim 0$, we approximate PG–metrics of type (7). In a particular case, we can use a conformal v–factor which is a 1–solitonic one, i.e. $\breve{\omega} \to \omega(r,t) = 4\arctan e^{q\sigma(r-vt)+q_0}$, where $\sigma^2 = (1-v^2)^{-1}$ and constants q, q_0, v, defines a 1–soliton solution of the sine–Gordon equation $\omega^{**} - \omega^{\bullet\bullet} + \sin\omega = 0$. Such a soliton propagates in time along the radial coordinate.

We now consider a reconstruction mechanism with distinguished off–diagonal cosmological effects [9] by generalizing some methods elaborated for $f(R)$ gravity in [2]. Any cosmological solution in massive, MGT and/or GR parameterized in a form (9) (in particular, as (20) and (22)) can be encoded into an effective functional $\widehat{f} - \frac{\mathring{\mu}^2}{4}\mathcal{U} = f(\widehat{R}), \widehat{R}_{|\widehat{\mathbf{D}}\to\nabla} = R$ (3). This allows us to work as in MGT; the conditions $\partial_\alpha f(\widehat{R}) = 0$ if $\widehat{R} = const$ simplify substantially the computations. The starting point is to consider a prime flat FLRW like metric

$$ds^2 = a^2(t)[(dx^1)^2 + (dx^2)^2 + (dy^3)^2] - dt^2,$$

where t is the cosmological time. In order to extract a monotonically expanding and periodic cosmological scenario, we parameterize $\ln|a(t)| = H_0 t + \tilde{a}(t)$ for a periodic function $\tilde{a}(t+\tau) = \ {}^1a\cos(2\pi t/\tau)$, where $0 < \ {}^1a < H_0$. Our goal is to prove that such a behavior is encoded into off–diagonal solutions of type (20)–(22).

We write FLRW like equations with respect to N–adapted (moving) frames (11) for a generalized Hubble function H,

$$3H^2 = 8\pi\rho \text{ and } 3H^2 + 2\mathbf{e}_4 H = -8\pi p.$$

Using variables with $\partial_\alpha f(\widehat{R})_{|\widehat{R}=const} = 0$, we can consider a function $H(t)$ when $\mathbf{e}_4 H = \partial_t H = H^*$. The energy–density and pressure of an effective

perfect fluid are computed

$$
\begin{aligned}
\rho &= (8\pi)^{-1}[(\partial_R f)^{-1}(\frac{1}{2}f(R) + 3H\mathbf{e}_4(\partial_R f)) - 3\mathbf{e}_4 H] \qquad (23)\\
&= (8\pi)^{-1}[\partial_{\widehat{R}} \ln \sqrt{|\widehat{f}|} - 3H^*] = (8\pi)^{-1}[\partial_{\widehat{R}} \ln \sqrt{|\frac{\mathring{\mu}^2}{4}\mathcal{U} + f(\widehat{R})|} - 3H^*],\\
p &= -(8\pi)^{-1}[(\partial_R f)^{-1}(\frac{1}{2}f(R) + 2H\mathbf{e}_4(\partial_R f) + \mathbf{e}_4\mathbf{e}_4(\partial_R f)) + \mathbf{e}_4 H]\\
&= (8\pi)^{-1}[\partial_{\widehat{R}} \ln \sqrt{|\widehat{f}|} + H^*] = -(8\pi)^{-1}[\partial_{\widehat{R}} \ln \sqrt{|\frac{\mathring{\mu}^2}{4}\mathcal{U} + f(\widehat{R})|} + H^*].
\end{aligned}
$$

In N–adapted variables, the equation of state, EoS, parameter for the effective dark fluid is defined by

$$
w = \frac{p}{\rho} = \frac{\widehat{f} + 2H^*\partial_{\widehat{R}}\widehat{f}}{\widehat{f} - 6H^*\partial_{\widehat{R}}\widehat{f}} = \frac{\frac{\mathring{\mu}^2}{4}\mathcal{U} + f(\widehat{R}) + 2H^*\partial_{\widehat{R}}\widehat{f}}{\frac{\mathring{\mu}^2}{4}\mathcal{U} + f(\widehat{R}) - 6H^*\partial_{\widehat{R}}\widehat{f}}, \qquad (24)
$$

when the corresponding EoS is $p = -\rho - (2\pi)^{-1}H^*$ and $\mathcal{U}(t)$ is computed, for simplicity, for a configuration of "target" Stückelberg fields $\phi^{\mu'} = \mathbf{e}^{\mu'}{}_{\underline{\mu}}\,\phi^{\underline{\mu}}$ when a found solution is finally modelled by generating functions with dependencies on t.

Taking a generating Hubble parameter $H(t) = H_0 t + H_1 \sin\omega t$, for $\omega = 2\pi/\tau$, we can recover the modified action for oscillations of off–diagonal (massive) universe (see similar details in [2]),

$$
f(R(t)) = 6\omega H_1 \int dt[\omega\sin\omega t - 4\cos\omega t(H_0 + H_1\sin\omega t)]\exp[H_0 t + \frac{H_1}{\omega}\sin\omega t]. \qquad (25)
$$

We can not invert analytically to find in explicit form R. Nevertheless, we can prescribe any values of constants H_0 and H_1 and of ω and compute effective dark energy and dark matter oscillating cosmology effects for any off–diagonal solution in massive gravity and/or effective MGT, GR. To extract contributions of $\mathring{\mu}$ we can fix, for instance, $\widehat{f}(\widehat{R}) = \widehat{R} = R$ and using (1) and (2) we can relate $f(R(t))$ and respective constants to certain observable data in cosmology.

The MGT theories studied in this work encode, for respective nonholonomic constraints, the ekpyrotic scenario which can be modelled similarly to $f(R)$ gravity. A scalar field is introduced into usual ekpyrotic models in order to reproduce a cyclic universe and such a property exists if we consider off–diagonal solutions with massive gravity terms and/or f–modifications. Let us consider a prime configuration with energy–density for pressureless matter $\mathring{\rho}_m$, for radiation and anisotropies we take respectively $\mathring{\rho}_r$ and $\mathring{\rho}_\sigma$ for radiation and anisotropies, κ is the spatial curvature of the universe and a target effective energy–density ρ (23). A FLRW model can be described by

$$
3H^2 = 8\pi[\frac{\mathring{\rho}_m}{a^3} + \frac{\mathring{\rho}_r}{a^4} + \frac{\mathring{\rho}_\sigma}{a^6} - \frac{\kappa}{a^2} + \rho].
$$

We generate an off–diagonal/massive gravity cosmological cyclic scenario containing a contracting phase by solving the initial problems if $w > 1$, see (24). A homogeneous and isotripic spatially flat universe is obtained when the scale factor tends to zero and the effective f–terms (massive gravity and off–diagonal contributions) dominate over the rests. In such cases, the results are similar to those in the inflationary scenario. For recovering (25), the ekpyrotic scenario takes place and mimic the observable universe for $t \sim \pi/2\omega$ in the effective EoS parameter $w \approx -1 + \sin \omega t / 3\omega H_1 \cos^2 \omega t \gg 1$. This allows us to conclude that in massive gravity and/or using off–diagonal interactions in GR cyclic universes can be reconstructed in such forms that the initial, flatness and/or horizon problems can be solved.

In the diversity of off–diagonal cosmological solutions which can constructed using above presented methods, there are cyclic ones with singularities of the type of big bang/ crunch behaviour. Choosing necessary types generating and integration functions, we can avoid singularities and elaborate models with smooth transition. Using the possibility to generate nonholonomically constrained f–models with equivalence to certain classes of solutions in massive gravity and/or off–diagonal configurations in GR, we can study in this context, following methods in [2, 9], , big and/or little rip cosmology models, when the phantom energy–density is modelled by off-diagonal interactions. We omit such considerations in this letter.

In summary, we have found new cosmological off–diagonal solutions in massive gravity with flat, open and closed spatial geometries. We applied a geometric techniques for decoupling the field equations and constructing exact solutions in $f(R)$ gravity, theories with nontrivial torsion and noholonomic constraints to GR and possible extensions on (co) tangent Lorentz bundles. A very important property of such generalized classes of solutions is that they depend, in principle, on all spacetime coordinates via generating and integration functions and constants. After some classes of solutions were constructed in general form, we can impose at the end nonholonomic constraints, cosmological approximations, extract configurations with a prescribed spacetime symmetry, consider asymptotic conditions etc. Thus, our solutions can be used not only for elaborating homogeneous and isotropic cosmological models with arbitrary spatial curvature, but also for study "non–spherical" collapse models of the formation of cosmic structure such as stars and galaxies (see also [8]).

Deriving cosmological scenarios only for diagonalizable metrics, there are possibilities to discriminate the massive gravity theory form the f–gravity and/or GR. Mathematically, we work with certain nonlinear systems of ordinary differential equations with general solutions depending on integration constants. Following such an approach, we positively have to modify the GR theory in order to explain observational data in modern cosmology and elaborate realistic quantum models of massive gravity. Fundamental field equations in gravity consists from nonlinear off–diagonal systems of PDE.

It is very difficult to construct off–diagonal solutions and to provide certain physical meaning for such spacetimes. Perhaps, such solutions are important at large cosmological scales because of the possibility to models polarization of constants and nonlinear interactions, anisotropies and generalized sources in modified gravity theories. A new and important feature is that the off–diagonal anisotropic configurations allows us to model cosmic accelerations and massive gravity and/or dark energy and dark matter effects as certain effective Einstein spaces. For a large class of such solutions, we can put the question: May be we do not need to consider modifications of the Einstein gravity but only to extend the constructions to off–diagonal solutions and nonholonomic systems and try to apply this in modern cosmology? This is a quite complicated theoretical and experimental problem and the main goal of this paper was to analyze such constructions from the viewpoint of massive gravity theory when off–diagonal effects can be alternatively explained to other types of gravity theories.

We also developed the reconstruction method for the massive gravity theory which admits and effective off–diagonal interpretation in GR and f–modified gravity with cyclic and ekpyrotic universe solution. The expansion is around the GR action if we admit a nontrivial effective torsion. For zero torsion constraints, it is also possible to perform off–diagonal cosmological models keeping the constructions in the framework of the GR theory. Our results indicate that theories with massive gravitons and off–diagonal interactions may lead to more complicated cyclic universe. Following such an approach, the ekpyrotic (little rip) scenario can be realized with no need to introduce an additional field (or modifying gravity) but only in terms of massive gravity or GR. Further constructions can be related to reconstruction scenarios of $f(R)$ and massive gravity theories leading to little rip universe. The dark energy for little rip models presents an example of non–singular phantom cosmology. Finally, we note that other types of non–singular super–accelerating universe may be also reconstructed in $f(R)$ gravity.

Acknowledgments: The work is partially supported by the Program IDEI, PN-II-ID-PCE-2011-3-0256. The author is grateful to organizers and participants of respective sections at MG13, where some results of this paper where presented. He also thanks P. Stavrinos, S. Hervik and S. Hassan for important discussions, critical remarks, or substantial support for seminars.

References

[1] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173 (1939) 211

[2] K. Hinterbichler, Rev. Mod. Phys. 84 (2012) 671; S. Nojiri, S. Odintsov and N. Shirai, arXiv: 1212.2079; S. Nojiri and S. Odintsov, Phys. Lett. B716 (2012) 377; S. Nojiri, S. Odintsov and D. Saez-Gomez, AIP Conf. Proc. 1458 (2011) 207, arXiv: 1108.0768

[3] H. van Dam and J. J. G. Veltman, Nucl. Phys B 22 (1970) 397; V. Zakharov, JETP Letters (Sov. Phys.) 12 (1970) 312; A. I. Vainshtein, Phys. Lett. B 39 (1972) 393

[4] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485 (2000) 2008; C. Deffayet, G. R. Dvali and G. Babadaze, Phys. Rev. D 65 (2002) 044023; M. A. Luty, M. Porrati and R. Rattazzi, JHEP 0309 (2003) 029; A. Nicolis and R. Rattazzi, JHEP 0406 (2004) 059

[5] D. G. Boulware and S. Deser, Phys. Rev. D 6 (1972) 3368; N. Arkani–Hamed, H. Gergi and M. D. Schwarz, Annals Phys. 305 (2003) 96

[6] C. de Rham, Phys. Lett. B 681 (2009) 89; C. de Rham and G. Gabadadze, Phys. Rev. D 82 (2010) 044020; C. de Rham, G. Gabadadze, and A. J. Tolley, Phys. Rev. Lett. **106** (2011) 231101; S. Hassan and R. A. Rosen, Phys. Rev. Lett. 108 (2012) 041101; JHEP 1204 (2012) 2012

[7] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava and A. J. Tolley, Phys. Rev. D 84 (2011) 124046; A. E. Gumrukcuoglu, C. Lin, and S. Mukoyama, JCAP 1203 (2012) 006; M. von Strauss, A. Schmidt–May, J. Enander, E. Mortsell and S. F. Hassan, JCAP 1203 (2012) 042; M. S. Volkov, JHEP 1201 (2012) 035; S. Koyama, G. Niz, and G. Tasinato, JHEP 1112 (2011) 065; P. Gratia, W. Hu and M. Wyman, Phys. Rev. D 86 (2012) 061504; Y. Akrami, T. S. Koivisto, M. Sandstad, JHEP 03 (2013) 099

[8] T. Kobayashi, S. Siino, M. Yamaguchi and D. Yoshida, Phys. Rev. D 86 (2012) 061505

[9] P. Stavrinos and S. Vacaru, Class. Quant. Grav. 30 (2013) 055012; S. Vacaru, Gen. Relat. Grav. 44 (2012) 1015; Europhys. Letters (EPL) 96 (2011) 5001; Class. Quant. Grav. 28 (2011) 215991; Int. J. Geom. Meth. Mod. Phys. 8 (2011) 9; arXiv: 1108.2022; Int. J. Theor. Phys. 49 (2010) 1788; Int. J. Mod. Phys. D 21 (2012) 1250072

[10] R. Geroch, J. Math. Phys. 12 (1971) 918; R. Geroch, J. Math. Phys. 12 (1972) 394; S. Vacaru, Int. J. Geom. Meth. Mod. Phys. 4 (2008) 1285

[11] L. Berezhiani, G. Chkareuli, C. de Rham, G. Gababadze and A. J. Tolley, Phys. Rev. D 85 (2012) 044024; K. Koyama, G. Niz and G. Tasinato, JHEP 1112 (2011) 065

[12] B. B. Kadomtsev and V. I. Petviashvili, Doklady Akademii Nauk SSS **192** (1970) 753; Sov. Phys. Dokl. **15** (1970) 539 [English translation]

[13] S. Vacaru, Acta Applicandae Mathematicae **110** (2010) 73